Numerical-Aided Elasto-Plastic Model for Circular Tunnel in Hoek-Brown Rock Masses

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ABSTRACT

This paper presents an elasto-plastic analytical solution for an axi-symmetrical circular tunnel. On account of the mathematical complexity, numerical treatments have been used to assist the elasto-plastic solution in evaluating the stress equilibrium, the strain compatibility equation, and the radius of yielding zone. In proposed model, rock mass obeys the latest non-linear generalized Hoek-Brown yield criterion, developed in 2002, in terms of its peak and residual strength parameters. The proposed approach considers a variable value for and residual strength parameter of rock mass, as governed by a strength loss parameter. The strength loss parameter makes it possible to model either elastic-perfectly plastic or elastic-brittle-plastic behaviour of the rock mass. The solution results in the analysis of the stress and strain state and the representation of the Ground Reaction Curve (GRC), which is commonly used in convergence-confinement method.

1 INTRODUCTION

The Preliminary design of an underground structure such as tunnels and shafts requires an analytical solution to predict the stress and strain state. An elasto-plastic solution makes it possible to determine the stresses, the displacements, and the radius of the plastic zone around the tunnel. For the assumption of the isotropy in field stress, circular shape of tunnel, homogeneity in rock mass, and axi-symmetrical plane strain condition, several approaches have been developed over the past 30 years, in which either Mohr-Coulomb (M-C) or Hoek-Brown (H-B) yield criterion was used.

Considering different models of rock behaviour, such as the elastic-perfectly plastic, elastic-brittle-plastic and elastic-strain softening, with different yield criteria M-C and H-B, the complexity of the solution differs.


Up to date, the most difficult part of deriving the equations was to obtain the plastic strain in plastic zone that necessitate many mathematical simplifications and treatments. More recently, researchers have tried to develop an elasto-plastic solution that satisfies the latest version of H-B yield criterion (Hoek et al. 2002) in which $a \geq 0.5$. Among them, Carranza-Torres (Carranza-Torres 2004) proposed a comprehensive closed-form solution using a transformation technique that made the solution more complex. Recently, Sharan (2007) pointed out the errors in the solutions by Brown et al. (1983) and Wang (1996) and developed also a closed form solution for $a \geq 0.5$. However, the author has found this solution to be incorrect.

The objective of this paper is to develop a numerical-aided elasto-plastic solution for the analysis of the stress and strain state around a circular tunnel in an elastic-perfectly plastic or elastic-brittle-plastic rock media obeying non-linear generalized Hoek-Brown yield criterion with $a \geq 0.5$. The proposed solution utilizes simple mathematical treatments to alleviate the complexity of the problem. The available mathematical softwares namely Mathematica (Matematica, Wolfram Research 2004) and Maple (Maple Inc 2003) are used to
solve the stress equilibrium and strain compatibility
equations. The resulting integration in product of the
strain compatibility equation is evaluated by so-called
Sympon’s rule. The radial stress at elastic-plastic
boundary around the tunnel is evaluated numerically
using the well-known Newton-Raphson Method. All
formulations of the solution can be implemented by a
programmable calculator for quick usage.

2 DEFINITION OF THE PROBLEM AND THE
MAIN ASSUMPTIONS

The problem is defined in Figure 1. Consider a deep
circular tunnel being excavated in an infinite medium
subjected to isotropic hydrostatic initial stress, \( P_0 \) (\( K=1 \)).
The excavation removes the boundary stresses around the
circumference of the tunnel, and the process may be
simulated by gradually reducing the internal (support)
pressure, \( P_i \). As \( P_i \) is reduced, a plastic zone is formed
when the material is overstressed, and the radial
displacement, \( u_r \), occurs. It is required to compute the
stresses and displacements around the tunnel, when plane
strain condition along the axis of the tunnel is reached.

The assumptions of homogeneity, isotropy, time
independency, and linear elasticity prior to failure of the
rock mass are made. The rock mass strength is assumed
by GSI (Geological Strength Index) (Hoek et al., 1997). This index lies in a range of 5-85 and can be
associated with the Geological Strength Index (GSI),
which characterizes the rock mass (Hoek, 1994; Hoek &
Brown, 1997). This index lies in a range of 5-85 and can be
quantified from available qualitative or quantitative
charts based on the degree of jointing of the rock
structure and the condition of the discontinuities. The
constants \( m_h \), \( s \) and \( a \) of Equation 1 are in turn obtained
by GSI (Geological Strength Index) (Hoek et al., 2002):

\[
\sigma'_i = \sigma'_s + \sigma'_e \left( m_h \frac{\sigma'_s}{\sigma'_e} + s \right)^a \\
\]

The coefficients \( m_h \), \( s \) and \( a \) in Equation 1 are semi-
empirical parameters. In practice, these parameters are
associated with the Geological Strength Index (GSI),
which characterizes the rock mass (Hoek, 1994; Hoek &
Brown, 1997). This index lies in a range of 5-85 and can be
quantified from available qualitative or quantitative
charts based on the degree of jointing of the rock
structure and the condition of the discontinuities. The
constants \( m_h \), \( s \) and \( a \) of Equation 1 are in turn obtained
by GSI (Geological Strength Index) (Hoek et al., 2002):

\[
m_h = m_i \exp \left( \frac{GSI - 100}{28 - 14D} \right) \\
s = \exp \left( \frac{GSI - 100}{9 - 3D} \right) \\
a = \frac{1}{2} + \frac{1}{6} \left( e^{GSI/15} - e^{-20/3} \right) \\
\]

In Equations 2 and 3, \( D \) is a factor that depends on the
degree of disturbance to which the rock has been
subjected due to blast damage and stress relaxation. This
factor varies between 0 and 1 (Hoek et al., 2002). The
Hoek-Brown yield condition for post-peak (residual)
strength parameters, used for the yielded zone around the
evacuation can be rewritten as:

\[
\sigma_\theta = \sigma_\theta + \sigma'_e \left( m_h \frac{\sigma'_s}{\sigma'_e} + s' \right)^a \\
\]

where \( \sigma'_e \) is the residual strength of the intact rock, \( m'_h \),
\( s' \) and \( a' \) are residual strength parameters of Hoek-Brown
failure criterion. As it has been proved that the extension
of the broken zone relies on the residual value of the
intact rock strength (Hoek & Brown, 1980; Brown et al.,
1983; Indraratna & Kaiser, 1990a; Cundall et al., 2003;
Carranza-Torres, 2004), so the effect of the compressive
strength of the rock material should be included in the
form of the residual value as it loses its initial value due

3 SOLUTION METHOD

3.1 Stress analysis

For a solution of the elasto-plastic problem, the equation
of equilibrium, the compatibility condition, a stress-strain
relationship in the elastic field, a yield criterion, a plastic
potential, and a flow rule are required. The stresses and
displacements in the elastic region can be easily
determined by observing the continuity of radial stresses
and displacements at the elastic-plastic interface. The
solution within the plastic region will depend on the
assumption of (a) the yield criterion, (b) the use of an
associated or a non-associated flow rule, and (c) the
dilatancy angle \( \psi \).

Yield initiation is assumed to occur following a non-
linear Hoek-Brown failure criterion. In this elasto-plastic
solution, the latest version of the Hoek-Brown yield
criterion (introduced in 2002) has been chosen:

\[
\sigma'_i = \sigma'_s + \sigma'_e \left( m_h \frac{\sigma'_s}{\sigma'_e} + s \right)^a \\
\]

2
to stress relief or an increase in the strain. A stress reduction scale should, therefore, be considered as:

\[ \sigma'_{ci} = S_r \sigma_{ci} \]  

(6)

where \( S_r \) refers to the strength loss parameter that quantifies the jump in strength from the intact condition to the residual condition or a measure of the degree of loss in strength that occurs immediately after the peak strength is reached. The parameter \( S_r \) characterizes the brittleness of the rock material: ductile, softening, or brittle. By definition, \( S_r \) will fall within the range \( 0 \leq S_r \leq 1 \) where \( S_r = 1 \) implies no loss in strength and the rock material is ductile, or perfectly plastic. On the other hand, if \( S_r = 0 \), the rock is brittle (elastic-perfectly brittle plastic) with the minimum possible value for the residual strength.

The combination of the stress equilibrium equation and Hoek-Brown failure criterion (Equation 5) results in a non-linear differential equation for the determination of the stress in the plastic (broken) zone around the tunnel:

\[ \frac{d\sigma_r}{dr} = \frac{\sigma'_{ci} (m_b \sigma_{ci} + s')^{a'}}{\sigma_{ci}'} = 0 \]  

(7)

The solution of the above differential equation is obtained, taking into account the boundary condition (at \( r=r_i \) , \( \sigma_r = 0 \)):

\[ \sigma_r = \frac{\Gamma - S'}{m_b} \sigma'_{ci} \]  

(8a)

\[ \Gamma = \left[ S'^{1-a'} - m_b(a' - 1) \ln \left( \frac{r}{r_i} \right) \right]^{\frac{1}{1-a'}} \]  

(8b)

Continuity of radial stress through the whole rock medium is assumed for the determination of the plastic zone radius. The radial stress at the elastic-plastic interface can be considered as a fictitious internal pressure for the outer elastic zone. In the pure elastic zone, the stress distributions are determined using so-called Lame’s solution. Hence, the following non-linear equation must be solved to determine the plastic zone radius. This approach has already been discussed by Brown et al. (1983), Wang (1996), Osgou (2007), and Sharan (2008).

\[ 2(P_o - \sigma_{re}) = \sigma_{ci} \left( m_b \frac{\sigma_{re}}{\sigma_{ci}} + s \right)^a \]  

(9)

An exact solution is only possible when \( a = 0.5 \) as determined by Brown et al. (1983), Sharan (2003, 2005), Park & Kim (2006), and Park et al. (2008):

\[ \sigma_{re, exact} = P_o + \frac{m_b \sigma_{ci}}{8} \]  

\[ \frac{1}{8} \sqrt{16P_m \sigma_{ci} + m_b^2 \sigma_{ci}^2 + 16 \sigma_{ci}^2 s} \]  

(10)

The negative sign in the above equation is acceptable and after abbreviating:

\[ \sigma_{re, exact} = P_o - M \sigma_{ci} \]  

(11a)

where:

\[ M = \frac{1}{2} \left( \frac{m_b}{4} + m_b \frac{P_o}{\sigma_{ci} s} \right)^{\frac{1}{2}} - \frac{m_b}{8} \]  

(11b)

On the other hand, a numerical technique; namely, the Newton-Raphson method (Press et al.2007), can be applied to approximate the exact solution of Equation 9 (Osgou, 2006). If solved for \( a \geq 0 \), \( \sigma_{re} \) is numerically calculated:

\[ 2(P_o - \sigma_{re}) = \sigma_{ci} \left( m_b \frac{\sigma_{re}}{\sigma_{ci}} + s \right)^a \]  

(12)

By equating the radial stresses at the elastic-plastic interface, determined from both the elastic and plastic sides, the plastic zone radius \( r_e \) can numerically be determined by assuming continuity of radial stress at the elastic-plastic boundary. It is also assumed that the field boundaries are far enough from the tunnel such that their influence on the solution for \( r_e \) is negligible.

Equating \( \sigma_r \) of Equation 8a (for \( \sigma_r \) at \( r=r_i \)) and \( \sigma_{re} \) of Equation 12, the normalized plastic zone radius can be derived as follows:

\[ r_e = r_i \epsilon^X \]  

(13a)

\[ X = \left[ S' \left( 1-a' \right) - \left( \frac{\sigma_{re} \cdot m_b}{\sigma_{ci}} + s \right)^{1-a'} \right]^{\left(1-a'\right)} \frac{m_b}{m_b'(a'-1)} \]  

(13b)

3.2 Strain analysis

Under the axi-symmetric plane strain condition, the strains and the displacements are expressed as (Timoshenko & Goodier, 1970):

\[ u_r = u_r(r), u_\theta = 0, u_z = 0 \]  

(14)

\[ \epsilon_r = \frac{du_r}{dr}, \epsilon_\theta = \frac{u_r}{r}, \epsilon_z = 0 \]  

(15)

where the subscripts \( r, \theta \), and \( z \) denote the radial, tangential, and longitudinal (axial) directions, respectively.
The compatibility condition is given by (Timoshenko & Goodier, 1970):

\[ \frac{d\varepsilon^p_\theta}{dr} + \frac{\varepsilon^e_\theta - \varepsilon^p_\theta}{r} = 0 \]  \hspace{1cm} (16)

Strains in elastic zone

Hooke’s law is applied to determine the radial and tangential strains in the elastic region surrounding the plastic zone for a plane strain condition (Timoshenko & Goodier, 1970):

\[ \sigma_j = C_{ijkl}\varepsilon_{kl} \]  \hspace{1cm} \text{for } i, j, k, l = 1, 2, 3 \]  \hspace{1cm} (17a)

\[ \varepsilon_r = \frac{1}{E} \left[ \sigma_r - \frac{v}{1-v} \sigma_\theta \right] \]  \hspace{1cm} (17b)

\[ \varepsilon_\theta = \frac{1}{E} \left[ \sigma_\theta - \frac{v}{1-v} \sigma_r \right] \]

where the 81 components \( C_{ijkl} \) of the fourth-rank tensor \( C \) are material constants (stiffness matrix).

Substituting stresses in elastic zone, obtained by Lame’s solution, into Equation 17 provides the strain field for plane strain condition:

\[ \varepsilon_r = \frac{(1-2v)P_o + (\sigma_{nk} - P_o) (r_g)}{2G} \frac{(r_g)^2}{r} \]  \hspace{1cm} (18a)

\[ \varepsilon_\theta = \frac{(1-2v)P_o + (\sigma_{nk} - P_o) (r_g)}{2G} \frac{(r_g)^2}{r} \]  \hspace{1cm} (18b)

Strains in plastic zone

For small deformation and infinitesimal strains, the total strains in the plastic zone are the sum of the elastic and plastic components:

\[ \varepsilon^e = \varepsilon^e + \varepsilon^p \]  \hspace{1cm} (19a)

or in the polar coordinate:

\[ \varepsilon^p_r = \varepsilon^p_r + \varepsilon^p_\theta \]

\[ \varepsilon^p_\theta = \varepsilon^p_r + \varepsilon^p_\theta \]  \hspace{1cm} (19b)

where the superscripts \( e \) and \( p \) denote the elastic and plastic components, respectively. Hooke’s law and flow rule have been applied to calculate the elastic and plastic strains, respectively. The elastic strains in the plastic zone are determined by substituting stresses in plastic zone into Hook’s constitutive laws.

The plastic strains in the plastic zone are, instead, governed by an appropriate flow rule postulated for the yielding behaviour. The flow rule of plasticity relating the plastic strain increment \( \varepsilon^p \) to the plastic potential \( Q \) is given by (Hill, 1950; Brown, 1986):

\[ \varepsilon^p = \lambda \frac{\partial Q}{\partial \sigma} \]  \hspace{1cm} (20)

Since the extent of yielding depends on the dilation characteristics of the failed rock, the flow rule must adopt the influence of dilation. In the present solution, a linear Mohr-Coulomb plastic potential has been adopted. Under a plane strain condition, the flow rule can be written as:

\[ d\varepsilon^p_r + N_\psi d\varepsilon^p_\theta = 0 \]  \hspace{1cm} (21)

where \( N_\psi = \frac{1 + \sin \psi}{1 - \sin \psi} = \tan^2 \left( \frac{45^\circ + \frac{\psi}{2}}{2} \right) \) and \( \psi \) is the dilatancy angle of the rock.

Having been determined the elastic strains in the plastic zone, the combination of strain compatibility (Equation 16) with the flow rule (Equation 21) gives rise to a solution for the strain field in the form of a non-linear differential equation:

\[ \frac{d\varepsilon^p_\theta}{dr} + \frac{\sigma_{\theta \psi}}{2Gr} \left[ s^{\psi - a} - m_\psi'(a' - 1) \ln \left( \frac{r}{r_g} \right) \right] \frac{\varepsilon^p_\psi}{\varepsilon^p_\theta} = \frac{1}{r} \frac{\sigma_{\theta \psi}}{2Gr} \left[ s^{\psi - a} - m_\psi'(a' - 1) \ln \left( \frac{r}{r_g} \right) \right] \]  \hspace{1cm} (22a)

The tangential strain at the elastic-plastic boundary (at \( r = r_g \)) produced by the reduction of \( \sigma_\psi \) to the plastic potential \( Q \) is (Brown et al. 1983; Sharan 2003, 2007; Park & Kim 2006):

\[ \varepsilon^p_\theta = \frac{(P_o - \sigma_{nk})}{2G} \]  \hspace{1cm} (22b)

Hence, the solution of Equation 22a is obtained using software Matematica (Wolfram Research 2004) or Maple (Maple Inc 2003) as:

\[ \varepsilon^p_r = \frac{(P_o - \sigma_{nk})}{2G} \int r^{N-1} \left[ s^{\psi - a} + m_\psi'(1 - a') \ln \left( \frac{r}{r_g} \right) \right] \frac{dr}{r} \]  \hspace{1cm} (23)

As can be observed from Equation 23, an integral function has been introduced into the result of the differential equation. The complete solution can be obtained provided that the integral on the right side of Equation 23 is evaluated numerically. Sympton’s rule is applied to approximately solve the integration (Waner & Costenoble, 2006).

The radial displacement field can finally be evaluated from any of the expressions of Equation 15, neglecting elastic strain due to its very small magnitude compared to...
the plastic strain \( \varepsilon_{pl} \ll \varepsilon_{pl}^0 \) and substituting \( r = r_0 \).

Therefore, the radial inward displacement of the tunnel surface can simply be determined as:

\[
\frac{u_r}{r_i} = \frac{1}{2G}\left[ \frac{(P_o - \sigma_{n0})}{r_i^{1/3}} + \frac{1}{r_i^{1/3}} \sigma_n (1 - \nu)(2 + a' \psi') \cdot \int \frac{\Gamma}{r_i^{1/3}} \Gamma \, dr \right]
\]

\(24\)

4 PRACTICAL APPLICATION OF THE PROPOSED MODEL

The following example, posed by Hoek & Brown (1980) and Carranza- Torres (2004), are intended to illustrate the practical application of the proposed solution and to compare the results of proposed model with those of Carranza-Torres’s solution. The geomechanical parameters of the rock mass used are given in Table 1.

A comparison between the proposed solution and that developed by Carranza-Torres (2004), in terms of stresses and displacements around the tunnel, has been made as demonstrated in Figures 2 and 3. As can be observed, a good agreement between both results is obtained. In spite of equality in radius of plastic zone \(5.09 \text{m}\) for both solutions, the radial displacement at tunnel boundary by Carranza-Torres’ solution are slightly higher than that of proposed elasto-plastic model. This can be attributed to the different ways in solving the strain compatibility equation resulting different values. The radial displacement at tunnel boundary are calculated 30.7 mm for proposed model and 34.1 mm for Carranza-Torres’ solution.

The proposed elasto-plastic solution makes it possible to depict the Ground Reaction Curve (GRC), which is main component of the Convergence-Confinement Method in tunnel design. The GRC for geomechanical parameters given in Table 1 is presented in Figure 4.

Table 1. Input parameters used in the practical example

<table>
<thead>
<tr>
<th>Geomechanical parameters</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tunnel radius ( r_i ) (m)</td>
<td>2</td>
</tr>
<tr>
<td>Far field stress ( P_o ) (MPa)</td>
<td>15</td>
</tr>
<tr>
<td>Deformation modulus ( E ) (GPa)</td>
<td>5.7</td>
</tr>
<tr>
<td>Poisson’s ratio ( \nu ) (-)</td>
<td>0.3</td>
</tr>
<tr>
<td>Hoek – Brown rock mass strength constant ( \alpha )</td>
<td>1.7</td>
</tr>
<tr>
<td>( m_b )</td>
<td>3.9E-3</td>
</tr>
<tr>
<td>( a' )</td>
<td>0.5</td>
</tr>
<tr>
<td>( \psi' )</td>
<td>0.85</td>
</tr>
<tr>
<td>( \psi )</td>
<td>1.9E-3</td>
</tr>
<tr>
<td>( \sigma_{ci} ) (MPa)</td>
<td>30</td>
</tr>
<tr>
<td>( \sigma_{ci}' ) (MPa)</td>
<td>27</td>
</tr>
<tr>
<td>Dilatancy angle ( \psi ) (°)</td>
<td>0 (non-associated)</td>
</tr>
<tr>
<td>Dilatancy Parameter ( N\psi )</td>
<td>1</td>
</tr>
</tbody>
</table>

5 CONCLUSIONS

A numerical-based elasto plastic solution for an axi-symmetrical circular tunnel in an isotropic and homogeneous medium that obeys generalized Hoek-Brown failure criterion was developed. Various numerical techniques have been adopted in order to solve the equilibrium and compatibility equations apart from taking advantage of available mathematical programs. The essential aim of proposed solution is intended to predict the stress and strain states and extension of yielding around tunnel subjected to a hydrostatic stress field. Furthermore, the proposed solution allows the representation of the Ground Reaction Curve (GRC),
which is considered as a practical means in tunnel support design. The proposed solution is also capable of predicting the ultimate tunnel convergence (at least two tunnel diameters behind the face), where three dimensional face effect are ignored. The practical application of the proposed solution was presented using a real example. A comparison between the results of the proposed solution and those of Carranza-Torres has been carried out and a good agreement was acquired. The proposed solution provides a practical means to quickly evaluate the deformaional behaviour of a tunnel. All calculation steps can simply be implemented by using a programmable calculator or spread sheet.

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**NOMENCLATURE**

GSI = Geological Strength Index

\( a \) = strength constant of Hoek-Brown failure criterion

\( a' \) = residual strength constant of Hoek-Brown failure criterion

\( C \) = cohesion of rock mass

\( D \) = disturbance factor of Hoek-Brown failure criterion

\( E \) = Young’s (elasticity) modulus

\( G \) = shear modulus

\( k \) = stress ratio

\( m_i \) = strength constant of Hoek-Brown failure criterion for intact rock

\( m_h \) = strength constant of Hoek-Brown failure criterion

\( m_i' \) = residual strength constant of Hoek-Brown failure criterion

\( m_h' \) = equivalent strength constant of Hoek-Brown failure criterion

\( N_{\psi} \) = dilation coefficient

\( P_i \) = fictitious support pressure

\( P_a \) = in-situ stress

\( Q \) = plastic potential

\( r \) = distance from tunnel center to point of interest

\( r_p \) = radius of plastic (broken, yielding) zone

\( r_t \) = tunnel radius

\( \delta_r \) = post-peak strength reduction factor

\( s \) = strength constant of Hoek-Brown failure criterion

\( s' \) = residual strength constant of Hoek-Brown failure criterion

\( s'' \) = equivalent strength constant of Hoek-Brown failure criterion

\( u_r \) = radial displacement

\( u_n \) = displacement at tunnel surface

\( u_l \) = longitudinal displacement

\( \gamma_r \) = rock mass unit weight

\( \gamma_{str} \) = shear strain in axi-symmetric problem

\( \varepsilon_{max} \) = maximum principal strain

\( \varepsilon_{min} \) = minimum principal strain

\( \varepsilon_r \) = radial strain

\( \varepsilon_\theta \) = tangential strain

\( \varepsilon_z \) = longitudinal strain

\( \varepsilon'' \) = elastic strain

\( \varepsilon''_p \) = plastic strain

\( \varepsilon''_t \) = total tangential strain

\( \varepsilon''_s \) = elastic tangential strain

\( \varepsilon''_b \) = plastic tangential strain

\( \varepsilon''_p \) = elastic radial strain

\( \varepsilon''_t \) = total radial strain

\( \dot{\varepsilon} \) = incremental plastic strain in flow rule

\( \lambda_1 \) = non-negative constant of proportionality in flow rule

\( \nu \) = Poisson’s ratio

\( \sigma \) = yield function

\( \sigma_1 \) = maximum principal stress

\( \sigma_3 \) = minimum principal stress

\( \sigma_{ul} \) = uniaxial compressive strength of intact rock

\( \sigma_{ul'} \) = residual compressive strength of intact rock

\( \sigma_r \) = radial stress

\( \sigma_\theta \) = tangential stress

\( \sigma_n \) = radial stress at elastic-plastic interface

\( \sigma_{re,exact} \) = exact solution of radial stress at elastic-plastic boundary

\( \sigma_{re,N} \) = approximate (numerical) solution of radial stress at elastic-
plastic boundary

\( \tau_{\theta \theta} \) = tangential stress at elastic-plastic interface

\( \phi \) = internal friction angle of rock

\( \psi \) = dilatancy angle of rock

Subscripts

\( r \) = radial

\( t \) = tangential

Superscripts

\( e \) = elastic

\( p \) = plastic

\( t \) = total

\( \Delta \) = increment